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Note

Rational and N-breather solutions for the 2D Toda lattice equation

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As is well known, one of the 2D Toda lattice equations is given as [1]

$$\left(\frac{\partial^2}{\partial t_1^2}+\frac{\partial^2}{\partial t_2^2}\right)r_n = 2e^{-r_n}-e^{-r_{n-1}}-e^{-r_{n+1}}. \tag{1}$$

In 1995, Vekslerchik [2] proved that r_n, given by the equations

$$e^{-r_n}=1+\epsilon|\psi_n|^2 \tag{2}$$

and

$$\epsilon=\pm 1, \tag{3}$$

satisfies (1), when ψ_n satisfies both the discrete nonlinear Schrödinger equation

$$i\frac{\partial\psi_n}{\partial t_1}=(1/2)(\psi_{n-1}+\psi_{n+1})\bigl(1+\epsilon|\psi_n|^2\bigr)-\psi_n \tag{4}$$

and the discrete complex MKdV equation

$$\frac{\partial\psi_n}{\partial t_2}=(1/2)(\psi_{n-1}-\psi_{n+1})\bigl(1+\epsilon|\psi_n|^2\bigr). \tag{5}$$

A more general relation existing among the 2D Toda lattice equation, the coupled nonlinear Schrödinger equations, and the coupled discrete complex MKdV equations was proved in 1997 by Hisakado [3].

In this note, using this relation, we present two groups of solutions for (1), i.e., rational and N-breather solutions.

We first present rational solution of (1).

For this purpose, we give an improved representation of an N-soliton solution of (1) presented in [2] and take its long-wavelength limits in the case of small N.

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We start the reformulation by separately deriving N-soliton solutions of the repulsive-type discrete nonlinear Schrödinger equation and the minus-type discrete MKdV equation from an N-soliton solution of the discrete Hirota equation given in [4]. Recombining them and using (2) in the case of $\epsilon = -1$ as well as (4) and (6) of [4] for constructing the solution, we find that an N-soliton solution of (1) is re-expressed as

$$e^{-r_n} = (1-\rho^2)\tau_{n-1}\tau_{n+1}/\tau_n^2, \tag{6}$$

$$\tau_n = \sum_{\mu=0,1} \exp\left(\sum_{i=1}^{N} \mu_i x_i + \sum_{i<j}^{(N)} A_{ij}\mu_i\mu_j\right), \tag{7}$$

in which

$$x_i = \kappa_i n + \omega_i^{(1)} t_1 + \omega_i^{(2)} t_2 + \delta_i, \tag{8}$$

$$\omega_i^{(1)} = -\rho^2 \cos\kappa_0 \sin\theta_i + (1-\rho^2)\sin\kappa_0 \sinh\kappa_i, \tag{9}$$

$$\omega_i^{(2)} = -\rho^2 \sin\kappa_0 \sin\theta_i - (1-\rho^2)\cos\kappa_0 \sinh\kappa_i, \tag{10}$$

and

$$\exp A_{ij} = \sinh\bigl[(\kappa_i-\kappa_j)/2\bigr]\sin\bigl[(\theta_i-\theta_j)/2\bigr]/\sinh\bigl[(\kappa_i+\kappa_j)/2\bigr]\sin\bigl[(\theta_i+\theta_j)/2\bigr]. \tag{11}$$

In (9)–(11), θ_i is given by the equation

$$\sin(\theta_i/2) = \epsilon_i|\rho|^{-1}(1-\rho^2)^{1/2}\sinh(\kappa_i/2), \tag{12}$$

where

$$\epsilon_i = (\pm)_i 1. \tag{13}$$

In this formula, ρ and κ_0 represent two fundamental parameters.

Rational solutions of (1) can be derived from this formula by using a procedure similar to one given in [5]. Two types of solutions exist:

(1) Assuming

$$N = 1 \tag{14}$$

and

$$e^{\delta_1} = -1, \tag{15}$$

we obtain

$$\tau_n = -\kappa_1 X + O\bigl(\kappa_1^2\bigr), \tag{16}$$

in which

$$\begin{aligned} X = n &- \bigl[\epsilon_1|\rho|(1-\rho^2)^{1/2}\cos\kappa_0 - (1-\rho^2)\sin\kappa_0\bigr]t_1 \\ &- \bigl[\epsilon_1|\rho|(1-\rho^2)^{1/2}\sin\kappa_0 + (1-\rho^2)\cos\kappa_0\bigr]t_2. \end{aligned} \tag{17}$$

(2) Assuming

$$N = 2, \tag{18}$$

$$e^{\delta_1} = (1/8)\rho^{-2}\bigl[1 - (4/3)\rho^2\bigr]\kappa_1\kappa_2 + (\kappa_1 + \kappa_2)/(\kappa_1 - \kappa_2), \tag{19}$$

$$e^{\delta_2} = (1/8)\rho^{-2}\bigl[1 - (4/3)\rho^2\bigr]\kappa_1\kappa_2 - (\kappa_1 + \kappa_2)/(\kappa_1 - \kappa_2), \tag{20}$$

and

$$\epsilon_2 = \epsilon_1, \tag{21}$$

we obtain

$$\begin{aligned} \tau_n = {}& -(1/6)\kappa_1\kappa_2(\kappa_1 + \kappa_2)\Bigl\{X^3 + (3/4)\rho^{-2}\bigl[1 - (4/3)\rho^2\bigr]X \\ & - \bigl\{\epsilon_1(3/2)|\rho|^{-1}\bigl[1 - (4/3)\rho^2\bigr](1 - \rho^2)^{1/2}\cos\kappa_0 + 2(1 - \rho^2)\sin\kappa_0\bigr\}t_1 \\ & - \bigl\{\epsilon_1(3/2)|\rho|^{-1}\bigl[1 - (4/3)\rho^2\bigr](1 - \rho^2)^{1/2}\sin\kappa_0 - 2(1 - \rho^2)\cos\kappa_0\bigr\}t_2\Bigr\} \\ & + O(\kappa^4). \end{aligned} \tag{22}$$

We next present an N-breather solution of (1).

Generalizing a 1-breather solution of the plus-type discrete Hirota equation presented in [4] into an N-breather solution, and using it in a derivation process analogous to the N-soliton case, we can find an N-breather solution of (1) given as follows:

$$e^{-r_n} = (1 + \rho^2)\tau'_{n-1}\tau'_{n+1}/\tau'^{2}_n, \tag{23}$$

$$\tau'_n = \sum_{\mu=0,1} \exp\left(\sum_{i=1}^{2N}\mu_i x_i + \sum_{i<j}^{(2N)} A_{ij}\mu_i\mu_j\right), \tag{24}$$

in which

$$x_i = \kappa_i n + \omega_i^{(1)} t_1 + \omega_i^{(2)} t_2 + \delta_i, \tag{25}$$

$$\omega_i^{(1)} = \rho^2\cos\kappa_0\sin\theta_i + (1 + \rho^2)\sin\kappa_0\sinh\kappa_i, \tag{26}$$

$$\omega_i^{(2)} = \rho^2\sin\kappa_0\sin\theta_i - (1 + \rho^2)\cos\kappa_0\sinh\kappa_i, \tag{27}$$

$$\sin(\theta_i/2) = \epsilon_i\,\mathrm{i}|\rho|^{-1}(1 + \rho^2)^{1/2}\sinh(\kappa_i/2), \tag{28}$$

$$\exp A_{ij} = \sinh\bigl[(\kappa_i - \kappa_j)/2\bigr]\sin\bigl[(\theta_i - \theta_j)/2\bigr]/\sinh\bigl[(\kappa_i + \kappa_j)/2\bigr]\sin\bigl[(\theta_i + \theta_j)/2\bigr] \tag{29}$$

for $i, j = 1, 2, \ldots, 2N$, and

$$\epsilon_i = 1 \quad \text{and} \quad \epsilon_{i+N} = -1, \tag{30}$$

$$\kappa_{i+N} = \kappa_i^*, \qquad \delta_{i+N} = \delta_i^* \tag{31}$$

for $i = 1, 2, \ldots, N$.

We have assumed that ρ and κ_0 are real parameters. We can easily check the real-valuedness of τ'_n by an inspection.

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